Supplementary lecture:

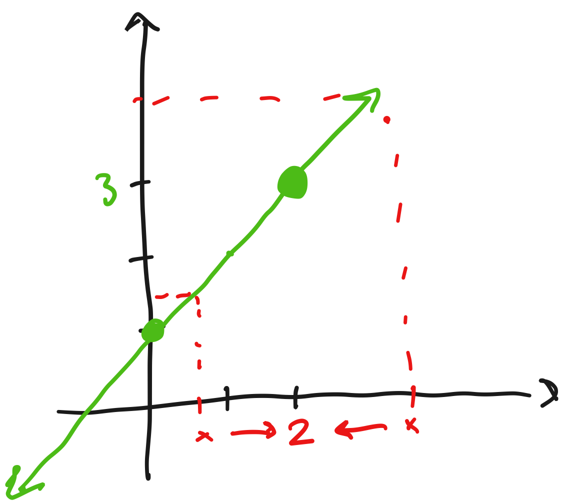
§2.2: Investigating Limits

§2.3: Limit Laws (start this section)

<u>Last time:</u>

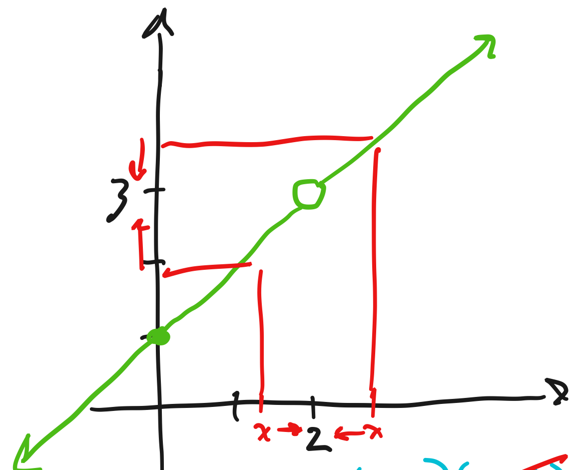
<u>Def</u> If $f(x) \to L$ as $x \to c$, then L is the <u>limit</u> of $f(x)$ as x approaches c.

We write: $\boxed{\lim_{x \to c} f(x) = L}$



$f(x) = x + 1$

$$\lim_{x \to 2} (x+1) = 3$$



$f(x) = \dfrac{(x+1)\cancel{(x-2)}}{\cancel{(x-2)}}$ if $x \neq 2$

$$\lim_{x \to 2} f(x) = 3$$

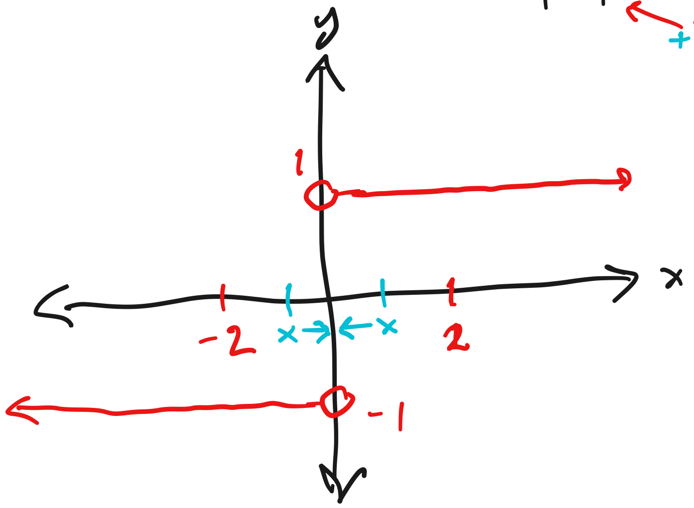
<u>Note</u> $\lim_{x \to c} f(x)$ does <u>not</u> depend on the value of $f(x)$ at c. In fact, $f(c)$ does not need to be defined.

<u>Terminology</u>: "$\dfrac{0}{0}$" is an <u>indeterminate form</u>

(more in §2.5)

Sometimes limits <u>don't</u> exist.

<u>Ex</u> (6): $f(x) = \dfrac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{at } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$



$\lim\limits_{x \to 0} f(x)$ <u>does not exist</u> (DNE)

is undefined

<u>Notation</u>:

$x \to c^+$ means x tends to c from the right (i.e. $x > c$)

$x \to c^-$ means x " " c " " left (i.e. $x < c$)

In the example,

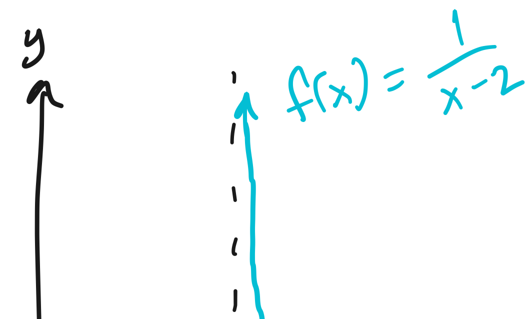
$\underbrace{\lim\limits_{x \to 0^+} f(x) = 1}_{\text{right-hand limit}}$, $\underbrace{\lim\limits_{x \to 0^-} f(x) = -1}_{\text{left-hand limit}}$

<u>Theorem</u> $\lim\limits_{x \to c} f(x) = L$ if and only if $\lim\limits_{x \to c^+} f(x) = L$

<u>and</u> $\lim\limits_{x \to c^-} f(x) = L$.

<u>Another difficulty</u>

<u>Ex</u> (8a) $f(x) = \dfrac{1}{x-2}$



$\lim_{x \to 2} \frac{1}{x-2}$ DNE

As $x \to 2^+$, $f(x)$ increases w/o bound

As $x \to 2^-$ $f(x)$ falls w/o bound

Q Does $\lim_{x \to 2^+} f(x)$ exist?

No! Would mean $\lim_{x \to 2^+} f(x) = L$ for some real number L

NOTE ∞ is not a real number

No, but still special

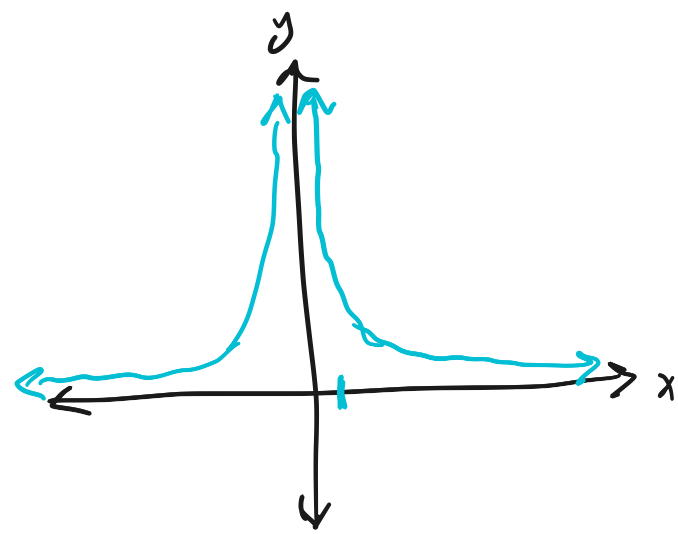
$\lim_{x \to 2^+} f(x) = \infty$

$\lim_{x \to 2^-} f(x) = -\infty$

lim "diverges to $+\infty$"

lim "diverges to $-\infty$"

Ex (8b) $f(x) = \frac{1}{x^2}$



As $x \to 0^+$ $f(x)$ increases w/o bound

Same for $x \to 0^-$

Write $\lim_{x \to 2} f(x) = +\infty$

$\lim_{x \to 2} f(x)$ "diverges to $+\infty$"

radians

Ex $f(x) = \sin\left(\frac{1}{x}\right)$

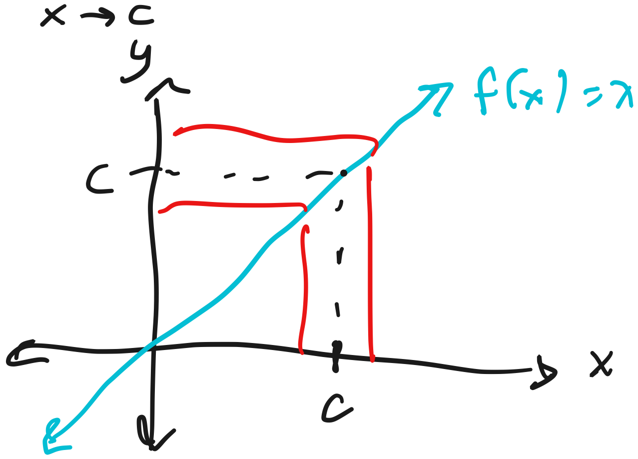
$\lim_{x \to 0} f(x)$ DNE

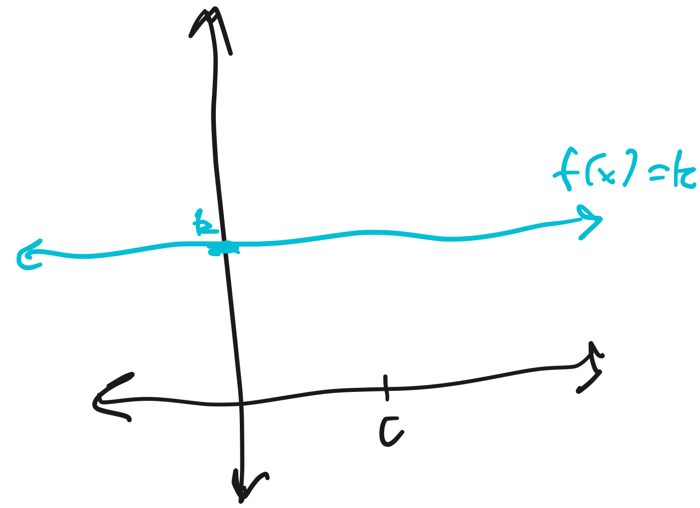
~~Done w/ §2.2~~

Basic limits (Thm 1 in §2.2) For any c,

$\lim_{x \to c} x = c$. And for any constant k,

$\lim_{x \to c} k = k$





§2.3 Basic Limit Laws

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then

1) $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$ (Sum Law)

2) For any constant k:

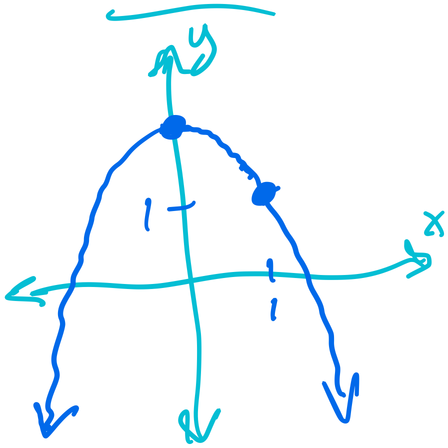
$\lim_{x \to c} [k \cdot f(x)] = k \cdot \lim_{x \to c} f(x)$ (constant multiple law)

3) $\lim_{x \to c} f(x)\, g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$ (Product law)

Ex $\lim_{x \to 1} (-x^2 + 2) = \lim_{x \to 1} (-x^2) + \lim_{x \to 1} (2)$ (Sum Law)

$= 2$

Check



$$= -1 \cdot \lim_{x\to 1} (\underbrace{x}_{f(x)} \cdot \underbrace{x}_{g(x)}) + 2 \quad \text{(const. mult. law)}$$

$$= -1 \cdot \lim_{x\to 1}(x) \cdot \lim_{x\to 1}(x) + 2 \quad \text{(prod. law)}$$

$$= -1 \cdot (1) \cdot (1) + 2 = \boxed{1}$$

Basic Limit Laws (cont'd)

4) $\displaystyle \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, assuming $\lim_{x\to c} g(x) \neq 0$

(Quotient Law)

5) For any constant r,

$$\lim_{x\to c} \big(\underset{\uparrow \atop -1}{f(x)}\big)^{\underset{\uparrow \atop 1/2}{r}} = \Big(\lim_{x\to c} f(x)\Big)^r, \text{ assuming } \Big(\lim_{x\to c} f(x)\Big)^r$$

is defined (and real)